

Exact information in $N=2$ theories

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This is intended to be a simple discussion of work done in collaboration with S. Cecotti, K. Intriligator and C. Vafa; and with H. Saleur. I discuss how $\text{Tr} (-1)^F F e^{-\beta H}$ can be computed exactly in any $N=2$ supersymmetric theory in two dimensions. It gives exact information on the soliton spectrum of the theory, and corresponds to the partition function of a single self-avoiding polymer looped once around a cylinder of radius β . It is independent of almost all deformations of the theory, and satisfies an exact differential equation as a function of β . For integrable theories it can also be computed from the exact S -matrix. This implies a highly non-trivial equivalence of a set of coupled integral equations with the classical sinh-Gordon and the affine Toda equations.

A fascinating example of supersymmetry is $N=2$ supersymmetry in two dimensions. Initially, its main use was as a world-sheet symmetry in string theory, but in recent years the subject has taken on a life of its own. One reason is that non-renormalization theorems often make calculations simpler and very elegant. As a result, one can often do exact calculations, even in models without conformal symmetry. Here I shall discuss the “index”

$$\text{tr } F(-1)^F e^{-\beta H}. \quad (1)$$

which is exactly calculable in any two-dimensional $N=2$ theory [1]. F is the fermion number, the conserved charge following from the $U(1)$ symmetry required in the $N=2$ algebra.

Aside from its intrinsic interest as a calculable non-perturbative quantity, there are a number of results which follow from the calculation of (1):

- 1) It gives detailed information on the exact soliton spectrum of the model.
- 2) It hints at still-unknown mathematical structure underlying $N=2$ models.
- 3) It provides a c -function on the space of $N=2$ theories.
- 4) The simplest $N=2$ model describes many properties of two-dimensional polymers (self-avoiding random walks). The index turns out to be the partition function for a single such polymer which loops around a cylinder [2], yielding the scaling function for the number of such configurations.
- 5) In the polymer “dense” phase, (1) has poles. These correspond to places where an excited-state energy level crosses the ground state [3]. Since perturbation theory generally breaks down after such an occurrence, this gives valuable quantitative information.
- 6) In any integrable $N=2$ theory, there are two different ways of doing the calculation, one giving a differential equation, and the other a set of coupled integral equations. In the simplest case, the differential equation is the well-known sinh-Gordon equation (a special case of Painlevé III); our result gives a previously-unknown set of integral equations with the same regular solutions.
- 7) Reversing the last item, it is the only known example where the thermodynamic Bethe ansatz equations [4,5] are equivalent to a differential equation.

To do these $N=2$ calculations, one does not need an infinite-dimensional symmetry: the calculation of (1) follows only from the finite-dimensional supersymmetry algebra. Most exact quantities follow from properties that are specifically two-dimensional like the infinite-dimensional conformal symmetry. Since this does not, there is no obvious reason

why similar exactly calculable quantities do not exist in higher dimensions. For example, one can write down a quantity in four-dimensional $N=2$ self-dual Yang-Mills theory which generalize many properties of (1). Unfortunately, we do not have the tools to calculate it, but we can hope that this is a technical obstacle and not something fundamental.

The details of these calculations are in the papers [1,2]. What I will try to do is describe first the setting: $N=2$ supersymmetric soliton theories in two dimensions. (These are not a restricted subclass: solitons are generic in off-critical $N=2$ theories.) Then I will discuss the two ways of deriving (1), skipping the details but hopefully making the methodology clear. I will present the results for a specific example, the simplest $N=2$ minimal model perturbed off the critical point. This is also convenient because this is the model which allows the calculation of the polymer quantities. I shall provide a few comments on this equivalence as well.

We first need to look at the $N=2$ symmetry algebra. In two dimensions there is the fermion-number charge F and four supercharges: the left movers Q_{\pm} and the right movers \overline{Q}_{\pm} . They obey

$$\begin{aligned}
Q_+^2 &= Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = \{Q_+, \overline{Q}_-\} = \{Q_-, \overline{Q}_+\} = 0 \\
\{Q_+, \overline{Q}_+\} &= 2\Delta & \{Q_-, \overline{Q}_-\} &= 2\Delta^* \\
\{Q_+, Q_-\} &= E - p & \{\overline{Q}_+, \overline{Q}_-\} &= E + p, \\
[F, Q_{\pm}] &= \pm Q_{\pm} & [F, \overline{Q}_{\pm}] &= \mp \overline{Q}_{\pm}
\end{aligned} \tag{2}$$

The crucial ingredient for us is the “central term” Δ , which is a c -number, and which depends only on the boundary conditions of the theory. That is, if the vacua of the theory are indexed by a , $\Delta = \Delta_{ab}$, where a and b are the vacua at left and right spatial infinity, respectively. One important result of this term is that the mass of a soliton m_{ab} connecting vacua a and b obeys a Bogomolny bound $m_{ab} \geq |\Delta_{ab}|$ [6]. Moreover, the representations of the supersymmetry algebra are special when this bound is saturated ($m = |\Delta|$). In this case, they are two-dimensional; otherwise, they are four-dimensional. The “reduced” representations play a special role in what follows.

First, I need to say what is meant by the word “index”. I put it in quotes because we do not know how to formulate (1) as a index in the precise mathematical sense. However, like a real one it is independent of many deformations of the theory. It is well known how Witten’s index $\text{tr} (-1)^F \exp(-\beta H)$ does not usually depend on β or on any finite deformations of the theory, because excitations with non-zero energy form fermion-boson

pairs whose contributions cancel [7]. This is actually not true in general: the arguments of [7] only apply in a box with periodic boundary conditions, which is not always possible with solitons. The fact that Witten's index in this situation can depend on β was discovered some time ago, and belongs to the class of “open-space index theorems” [8].

With the open space required for solitons, we can show that the index (1) depends only on “ F -term” perturbations of the action (including β); it does not depend on “ D -terms”. This F -term dependence is not a nuisance, it is the point: the differential equations for the index are in terms of the parameters which describe the F -terms. D -terms have all four supercharges operating on some field (in superspace this is an integration over $d^4\theta$), while F -terms have only Q_+ and \overline{Q}_+ ($d^2\theta^+$) or the complex conjugate Q_- and \overline{Q}_- ($d^2\theta^-$). The powerful theorems I mentioned earlier say that quantum effects renormalize F -terms only by their naive scaling dimension. In a Landau-Ginzburg theory, the D - and F -terms correspond to the kinetic term and the superpotential, respectively, so the theorems say that the form of the superpotential is not renormalized. The index is independent of D -term, so it does not get renormalized. It changes only under perturbation by relevant and marginal operators, and there are generically only a finite number of these. In a sigma model, deformations of the target-space metric which do not change the Kähler class do not change the F -term; in the case with target space a sphere (CP^1) this means that the index depends only on the area of the sphere.

To prove that our index does not depend on the D -terms requires only the fact that $\{(-1)^F, Q\} = 0$ for any Q , the relations (2), and the cyclic property of the trace. Any D -term variation of the action can be written as $\{Q_+, [\overline{Q}_-, \Lambda]\}$, where Λ is some field (in fact Λ is $Q_- \overline{Q}_+$ of something). Thus infinitesimally the change in (1) is

$$\begin{aligned}
\delta I &= \text{tr} F(-)^F \{Q_+, [\overline{Q}_-, \Lambda]\} e^{-\beta H} \\
&= \text{tr} F(-)^F (Q_+ [\overline{Q}_-, \Lambda] + [\overline{Q}_-, \Lambda] Q_+) e^{-\beta H} \\
&= \text{tr} (-)^F [F, Q_+] [\overline{Q}_-, \Lambda] e^{-\beta H} \\
&= \text{tr} (-)^F (Q_+ \overline{Q}_- \Lambda - Q_+ \Lambda \overline{Q}_-) e^{-\beta H} \\
&= \text{tr} (-)^F \{Q_+, \overline{Q}_-\} \Lambda e^{-\beta H} \\
&= 0 .
\end{aligned} \tag{3}$$

If you try a similar argument with an F -term variation, you find that the index does indeed change. Similarly, objects like $F^2(-1)^F$ do depend on the D -term. This argument is heuristic, because one must define quantum operators precisely to ensure the cyclic

property of the trace. One can also prove the D -term invariance by using the path integral; I leave it as an exercise to the reader to decide if this is more or less rigorous than the above argument.

Before computing the index exactly, it is first useful to calculate it in some simple limits. In the high-temperature (ultraviolet) limit and in fact at any critical point, its largest eigenvalue turns out to be proportional to the central charge of the conformal theory. (Notice that the index is actually a matrix, with rows and columns labelled by the different vacua, corresponding to the allowed boundary conditions at plus and minus spatial infinity.) Thus the index provides a c -function on the space of $N=2$ theories, although it is not the same as Zamolodchikov's [9]. Although in all known unitary examples the index decreases as one goes into the IR, we do not have a general proof of this.

For low temperature (the infrared), we can expand it in powers of $\exp(-m\beta)$, where m is any particle mass. This corresponds to an expansion in terms of particle states. It is easy to calculate the one-particle piece explicitly. We need only the density of states for a single particle, which follows from putting the particle in a box of length L and quantizing its momentum via $p = n\pi/L$. The density of states is $g(p) \equiv dn/dp = L/\pi$. The contribution of a particle with fermion number f and mass m to the index is

$$\begin{aligned} & f(-)^f \int_0^\infty dp \frac{L}{\pi} e^{-\beta\sqrt{p^2+m^2}} \\ &= f(-)^f \frac{L}{\pi} K_1(m\beta), \end{aligned} \tag{4}$$

where $K_1(x)$ is a Bessel function. We must not forget that the particles come in multiplets under the supersymmetry. For particles in the reduced doublet representation, the multiplet has charges $(f, f-1)$ and so the contribution is proportional to $f - (f-1) = 1$ and is

$$(-)^f \frac{L}{\pi} K_1(\Delta\beta).$$

I have written the mass m of this reduced multiplet as Δ to show that this contribution is purely global: it depends only on the boundary conditions. I would also like to point out that f is usually not an integer in these $N=2$ theories [10]; the phenomenon of fractional fermion number in soliton theories has been known for some time [11]. For a four-dimensional multiplet $(f+1, f, f, f-1)$, something interesting happens: the one-particle contribution vanishes! This is because $f+1 - 2f + f-1 = 0$. We see the first useful piece of information coming from the index: it counts the number of solitons in reduced multiplets.

Since one method of exact computation comes only from the $N=2$ chiral ring, this is a valuable way of finding the (doublet) spectrum of the theory. We also see how the index is a much simpler object than the full partition function, but still complex enough to give us non-trivial information.

One can calculate the two-particle contribution in this manner, but beyond this (except for the integrable theories to be discussed shortly) the calculation is too hard. One might think that all four-dimensional representations of supersymmetry do not contribute to the index, by the above argument. However, in the multi-particle case, the argument is not applicable. If space is a finite box, one cannot take periodic boundary conditions and keep the solitons. Any other boundary condition breaks supersymmetry, so the members of a multiplet do not necessarily cancel among one another. If space is the infinite line, one has a continuum of states. The densities for the members of a multiplet are not necessarily the same, so again no cancellation need happen. This is why Witten's index can depend on β in soliton theories. One can think of this as sort of an anomaly: by putting the theory in a box one breaks the supersymmetry, and the effect remains even as one takes the box boundaries to infinity.

We now discuss the methods for computing the index exactly. The first method uses the exact S -matrix of the solitons. This can be found in an integrable theory, which, luckily, covers many of the simplest and most interesting $N=2$ theories [12,10]. Understanding how to get an exact S -matrix is a long story, but the constraints of an integrable theory generally fix it uniquely: for a discussion see [13]. From the exact S -matrix, one can then use the thermodynamic Bethe ansatz to calculate the index [4,5]. This is a clever trick: one puts the particles in a box and quantizes the momenta like we did above for the one-particle contribution. Because the exact S -matrix is completely elastic (individual momenta do not change in a collision) the i th momentum is quantized via the condition ¹

$$e^{ip_i L} \prod_j S(p_i, p_j) = 1 ;$$

one can think of this as bring the i th particle around the world and scattering it off of all the others. If the scattering is trivial ($S=1$) we obtain the free-particle quantization condition $p_i = 2\pi n_i/L$. Using this quantization as a constraint, one minimizes the free

¹ This relation is for periodic boundary conditions, but is easily modified to the fixed case required for solitons.

energy to obtain thermodynamic quantities like (1). The answer is given in terms of non-linear integral equations. The calculation was done for many $N=2$ theories in [10]; I will give an example below.

The other method applies to any $N=2$ theory, and uses “topological-antitopological fusion” [14]. The input required here is the chiral ring coefficients $(C_i)_k^j$, which can be determined using topological field theory [15]. First, we define a “topological” state $|a\rangle$ on a disk by adding a term to the action

$$\int \frac{i}{2} j_\mu \omega^\mu.$$

This couples the fermion-number current j_μ to the spin connection of the manifold ω^μ . Basically, this amounts to putting a (Ramond) ground state a on the boundary of the disk. If we wanted to do topological field theory on the sphere, we would then glue another disk like this to it, and calculate the topological metric η_{ab} . This has all sorts of marvelous properties: for example, it is independent of the shape of the sphere. However, we are interested in information in the full theory, not the topological subsector. By putting the ground state $|a\rangle$ at one end of a cylinder of circumference β and long length L , and a conjugate state $|\bar{b}\rangle$ at the other end (this has fermion-number current coupled to *minus* half the spin connection), we calculate the ground-state metric $g_{a\bar{b}}$. This can be thought of as a generalization of Berry’s phase to a curvature on the multi-dimensional space of ground states; this was first discussed in quantum mechanics in [16]. In [14], a differential equation for $g_{a\bar{b}}$ was derived. The derivatives are in parameter space, which here governs the F -term variations (the relevant and marginal $N=2$ -preserving perturbations of the theory, which are in one-to-one correspondence with the ground states). Letting i and j index these variations, the (matrix) equation is

$$\bar{\partial}_j(g\partial_i g^{-1}) = \beta^2[C_i, gC_j^\dagger g^{-1}] \quad (5)$$

It is a simple calculation to derive that the index (1) is

$$I = iL(g\partial_\beta g^{-1} + \frac{n}{\beta}), \quad (6)$$

where n is the coefficient of the chiral anomaly (in a Landau-Ginzburg theory it is the number of superfields) and ∂_β means that we vary the inverse temperature β or equivalently the mass scale of the theory (since all quantities must depend on the dimensionless ratio $m\beta$). This should not be surprising: we have coupled fermion number to the geometry, so

when one varies β and changes the circumference of the cylinder, it brings down the F in front of $\text{tr} F(-)^F \exp(-\beta H)$.

We will display these results in the simplest $N=2$ model: the first member of the $N=2$ discrete series perturbed by the only relevant operator which preserves supersymmetry. This turns out to be the ordinary sine-Gordon model at a specific coupling ($\beta^2 = \frac{2}{3}8\pi$) where it is $N=2$ supersymmetric. In the Landau-Ginzburg picture this is described by a single superfield X (consisting of a complex boson ϕ and a Dirac fermion) with superpotential $W = X^3/3 - \lambda X$. The potential for ϕ is $|\partial W/\partial X|^2_{X=\phi}$, giving $(\phi^2 - \lambda)(\phi^{*2} - \lambda^*)$ (supersymmetric ϕ^4 with a double well). This has two minima, at $\phi = \pm\sqrt{\lambda}$. The entire spectrum of this theory turns out to be two doublets of solitons. One doublet has $\phi = -\sqrt{\lambda}$ at negative spatial infinity and $\phi = \sqrt{\lambda}$ at positive infinity, while the other goes in the opposite direction. Since the theory has a charge conjugation invariance the fermion numbers of a doublet must be $(1/2, -1/2)$. The model is integrable and the exact S -matrix for these solitons was found in [10].

We can now apply our two methods. The result of the first method (starting from the exact S -matrix) is that the index is

$$Q(z = m\beta) \equiv -i\frac{\beta}{L}I = z \int \frac{d\theta}{2\pi} \cosh \theta e^{-A(\theta; z)}. \quad (7)$$

where the function $A(\theta; z)$ obeys the integral equations

$$\begin{aligned} A(\theta; z) &= z \cosh \theta - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \ln(1 + B^2(\theta'; z)) \\ B(\theta; z) &= - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} e^{-A(\theta'; z)}. \end{aligned} \quad (8)$$

The topological-antitopological analysis of this theory was discussed at length in [14]. The metric on the space of ground states in the basis spanned by 1 and X can be written as $g = \beta e^{\sigma_3 u(z)/2}$. Using (5) with the chiral ring $X^2 = \lambda$, it follows that $u(z)$ satisfies the radial sinh-Gordon equation, (a special case of Painlevé III)

$$\frac{d^2 u}{dz^2} + \frac{1}{z} \frac{du}{dz} = \sinh u. \quad (9)$$

The index $Q(z)$ is

$$Q(z) = \frac{1}{2} z \frac{d}{dz} u(z). \quad (10)$$

The boundary conditions are fixed by demanding regular behavior at $mR \rightarrow \infty$, and have been discussed in detail in [17].

The equivalence between solutions of these integral equations and this differential equation was unknown and is very surprising, since both types of equation have been studied extensively. The sinh-Gordon equation of course is well known in mathematics and physics, while as a result of the thermodynamic Bethe ansatz, integral equations like (7) have been studied a great deal in the last several years. We lack a direct mathematical proof of the equivalence, although we have checked it numerically and in high- and low-temperature limits. We have a hint that a great deal of structure in these models is still undiscovered. I would also like to comment that for the remainder of the $N=2$ discrete series perturbed by the least-relevant operator and for the CP^1 sigma model, the index obeys the same differential equation, but with a slightly different boundary condition. The integral equations are modified slightly by adding a constant to the first of (8). If one perturbs the minimal model by the most-relevant operator, one obtains the Toda hierarchy of differential equations. One then can find integral equations equivalent to these as well.

The last thing I would like to discuss is the connection with polymers [2]. It turns out that many properties of 2d polymers can be described by studying this $N=2$ theory. Unfortunately, we lack a direct map of all quantities in the polymer theory to those in the $N=2$ theory; the equivalence is often quite subtle. However, we have shown that the index corresponds to the partition function of a single ring polymer (closed self-avoiding random walk) looped once around a cylinder of radius β . The parameter m is a scaled version of the distance from the polymer critical point. This critical point separates the high-temperature “dilute” phase from the low-temperature “dense” phase. In the dense phase, the polymers cover all of space. This corresponds to $(m\beta)^{4/3}$ negative, and the theory is non-unitary. The integral equations derived above diverge when continued to these values, but the differential equation can be continued, and becomes the cosh-Gordon equation. When one continues the solution appropriate in the dilute phase, one finds poles in the dense phase. In [3], we have argued in a number of ways that these poles correspond to level crossings. These are places where an excited-state energy reaches zero and crosses the high-temperature (supersymmetric) ground state. There are an infinite number of these crossings between the polymer critical point and the zero-temperature dense critical point. One can attempt to find an exact S -matrix for the dense phase as well, but because of the level crossings, the issue is very subtle.

There are a number of related developments:

- 1) Some new models of massless solitons [18] describe the flows from $c=3$ into an $N=2$ minimal model with central charge $3k/k + 2$. From the exact S -matrix, we calculate the index, and find that it does not change during this flow. Thus the flow is all D -term! This can be thought of as a flow in the Landau-Ginzburg model from a free superfield at $c=3$ to the (unknown) D -term which describes the minimal model. All along the way, the potential stays at X^{k+2} .
- 2) There is a similar but distinct object, the elliptic genus [19]

$$\text{tr } e^{i\alpha F_L} (-1)^{F_R} q^{L_0} \bar{q}^{\bar{L}_0},$$

which also does not depend on the D -term. As compared to our index, this has the advantage that one computes it on a torus (not just the cylinder) and gets a full character. The disadvantage is that it requires separate left and right fermion numbers, which usually does not happen off the critical point. Thus it is a function of β/L , while (1) is a function of $m\beta$.

- 3) Since $N=2$ supersymmetry is a simple case of the affine quantum-group symmetry underlying many two-dimensional models, one can hope that similar objects occur in non-supersymmetric two-dimensional models. Some preliminary steps in this direction were made in [20].

- 4) This work has been extended to show how to relate the soliton spectrum off the critical point to the charges of the chiral primary fields at the critical point [21]. This observation may be useful in classifying $N=2$ conformal field theories. In addition, similar objects have been discussed in [22]; these have deep geometrical and physical significance, and show that the uses of $N=2$ supersymmetry are far from being completely known!

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